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# NORMAL FORMS AND HOPF BIFURCATION FOR PARTIAL DIFFERENTIAL EQUATIONS WITH DELAYS

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Abstract. The paper addresses the computation of normal forms for some Partial Functional Differential Equations (PFDEs) near equilibria. The analysis is based on the theory previously developed for autonomous retarded Functional Differential Equations and on the existence of center (or other invariant) manifolds. As an illustration of this procedure, two examples of PFDEs where a Hopf singularity occurs on the center manifold are considered.

## 1. Introduction and preliminaries

The purpose of this paper is to compute normal forms on center manifolds (or other invariant manifolds) for Partial Functional Differential Equations (PFDEs) near equilibrium points, and use them to study the qualitative behavior of solutions on those manifolds, namely when a Hopf bifurcation occurs. It turns out that the coefficients of normal forms are explicitly given in terms of the coefficients of the original PFDE. In [9], a center manifold theory for reaction–diffusion equations with delays was developed and a coupled system of scalar ordinary differential equations as the equation on the center manifold was obtained. However, this system is given implicitly in terms of the considered PFDE. With the approach presented here, we give explicit normal forms (in the usual sense of Ordinary Differential Equations (ODEs)) for the equation giving the flow on the center manifold, without having to compute the manifold beforehand. In the particular case of generic Hopf bifurcation near equilibria, we show that, under certain conditions, that normal form coincides (up to third order terms) with the normal form for the FDE *associated* (in a precise and natural way) with the given PFDE. For partial differential equations with delays, we adopt the hypotheses in [9], as well as most of the notation, and we follow the work in [5], [6] for autonomous retarded Functional Differential Equations (FDEs).

In the following, $\Omega \subset \mathbf{R}^n$ is open, $X$ is a Hilbert space of functions from $\overline{\Omega}$ to $\mathbf{R}^m$ with inner product $\langle \cdot, \cdot \rangle$, and $\mathcal{C} = C([-r, 0]; X)$ $(r > 0)$ is the Banach space of continuous maps from $[-r, 0]$ to $X$ with the sup norm. Nevertheless, all the theory can be directly applied to the case where the functions in $X$ have values in $\mathbf{C}^m$. We write $u_t \in \mathcal{C}$ for $u_t(\theta) = u(t+\theta), -r \leq \theta \leq 0$ (see [8] for standard notation and results about FDEs). We consider PFDEs with an equilibrium point at the origin,

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given in abstract form (i.e., in the phase space $\mathcal{C}$) as

$$\frac{d}{dt}u(t) = d\Delta u(t) + L(u_t) + F(u_t) \qquad (t > 0), \tag{1.1}$$

where $d > 0$, $\mathrm{dom}(\Delta) \subset X$, $L \in \mathcal{L}(\mathcal{C}; X)$, i.e., $L : \mathcal{C} \to X$ is a bounded linear operator, and $F : \mathcal{C} \to X$ is a $C^k$ function ($k \geq 2$) such that $F(0) = 0, DF(0) = 0$. As an example, we consider the Hutchinson equation with diffusion in Section 5.

For the linearized equation about the equilibrium zero, $\frac{d}{dt}u(t) = d\Delta u(t) + L(u_t)$, we assume hypotheses (H1)-(H4) in [9] (see also [11]):

(H1) $d\Delta$ generates a $C_0$ semigroup $\{T(t)\}_{t\geq 0}$ on $X$ with $|T(t)| \leq Me^{\omega t}$ (for some $M \geq 1, \omega \in \mathbb{R}$) for all $t \geq 0$, and $T(t)$ is a compact operator for $t > 0$;

(H2) the eigenfunctions $\{\beta_k\}_{k=1}^{\infty}$ of $d\Delta$, with corresponding eigenvalues $\{\mu_k\}_{k=1}^{\infty}$, form an orthonormal basis for $X$, $\mu_k \to -\infty$;

(H3) the subspaces $\mathcal{B}_k := \{\langle v(\cdot), \beta_k\rangle \beta_k \,|\, v \in \mathcal{C}\}$ of $\mathcal{C}$ satisfy $L(\mathcal{B}_k) \subset \mathrm{span}\{\beta_k\}$;

(H4) $L$ can be extended to a bounded linear operator from $B\mathcal{C}$ to $X$, where $B\mathcal{C} = \{\psi : [-r, 0] \longrightarrow X \,|\psi$ is continuous on $[-r, 0), \exists \lim_{\theta\to 0^-} \psi(\theta) \in X\}$, with the sup norm.

Under (H1) and (H2), it was shown in [13] that the initial value problem

$$\begin{aligned} u(t) &= T(t)\phi(0) + \int_0^t T(t-s)L(u_s)ds, \quad t \geq 0,\\ u_0 &= \phi \in \mathcal{C}, \end{aligned}$$

has a unique continuous solution $u(t;\phi)$ for $t \geq -r$, and $\{W(t)\}_{t\geq 0}, W(t)\phi = u_t(\cdot;\phi)$, is a $C_0$–semigroup of linear (and compact for $t > r$) operators on $\mathcal{C}$, with infinitesimal generator $A$ given by

$$(A\phi)(\theta) = \dot{\phi}(\theta),\ \mathrm{dom}(A) = \{\phi \in \mathcal{C} : \dot{\phi} \in \mathcal{C}, \phi(0) \in \mathrm{dom}(\Delta), \dot{\phi}(0) = d\Delta\phi(0) + L\phi\}.$$

$A$ has only its point spectrum, and $\sigma(A) = \sigma_P(A) = \{\lambda \in \mathbf{C} : \mathbf{\Delta}(\lambda)y = 0$, for some $y \in \mathrm{dom}(\Delta) \setminus \{0\}\}$, where

$$\mathbf{\Delta}(\lambda)y = \lambda y - d\Delta y - L(e^{\lambda\cdot}y)\ . \tag{1.2}$$

For any $a \in \mathbf{R}$, the number of solutions of (1.2) such that Re $\lambda \geq a$ is finite. Using the decomposition of $X$ by $\{\beta_k\}_{k=1}^{\infty}$ and (H3), equation $\mathbf{\Delta}(\lambda)y = 0$ is equivalent to the sequence of "characteristic" equations

$$\lambda\beta_k - \mu_k\beta_k - L(e^{\lambda\cdot}\beta_k) = 0 \quad (k \in \mathbf{N}), \tag{$1.3_k$}$$

and there exists an $N$ such that all the solutions of ($1.3_k$) satisfy Re $\lambda < 0$ for $k > N$ (cf. [9], [11], [13]). Under (H4), it was shown in [11] that the solutions of (1.1) with initial conditions $\phi \in \mathcal{C}$ in the integral formulation above satisfy the variation of constants formula

$$u_t = W(t)\phi + \int_0^t W(t-s)X_0F(u_s)ds, \quad t \geq 0\ , \tag{1.4}$$

where

$$X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0,\\ I, & \theta = 0. \end{cases}$$

In [9], the existence of a local center manifold (invariant under the semiflow defined by (1.4)) was proved. Its dimension is equal to the number of $\lambda \in \sigma(A)$ with real part zero, counting multiplicities.

Let $\Lambda$ be the finite set $\Lambda = \{\lambda \in \sigma(A)\ : \operatorname{Re}\lambda = 0\}$, or another finite subset of $\sigma(A)$ with which an invariant manifold is associated (e.g., in [11] results on the unstable manifold are stated, corresponding to the choice $\Lambda = \{\lambda \in \sigma(A)\ : \operatorname{Re}\lambda > 0\}$), and suppose $\Lambda \neq \emptyset$. Let $C := C([-r,0];\mathbf{R})$ and, for each $k \in \mathbf{N}$, define $L_k : C \longrightarrow \mathbf{R}$ by

$$L_k(\psi)\beta_k = L(\psi\beta_k)\ . \tag{$1.5_k$}$$

Then, on $\mathcal{B}_k$, the linear equation $\frac{d}{dt}u(t) = d\Delta u(t) + L(u_t)$ is equivalent to the FDE on $\mathbf{R}$

$$\dot{z}(t) = \mu_k z(t) + L_k z_t\ , \tag{$1.6_k$}$$

with characteristic equation given by ($1.3_k$). Throughout this paper, unless otherwise stated, we assume that $\Lambda = \{\lambda \in \sigma(A)\ : \operatorname{Re}\lambda = 0\}$ and that the basis $\{\beta_k\}_{k=1}^{\infty}$ is ordered in such a way that $\Lambda$ contains exactly the solutions on the imaginary axis of the first $N$ equations ($1.3_k$): $\Lambda = \{\lambda \in \mathbf{C}\ :\ \lambda$ is a solution of ($1.3_k$) with $\operatorname{Re}\lambda = 0$, for some $k \in \{1,\dots,N\}\}$. For $1 \le k \le N$, defining $\eta_k \in BV([-r,0];\mathbf{R})$ such that

$$\mu_k\psi(0) + L_k\psi = \int_{-r}^{0} d\eta_k(\theta)\psi(\theta), \quad \psi \in C\ , \tag{$1.7_k$}$$

and $(\cdot,\cdot)_k$ the adjoint bilinear form on $C^* \times C, C^* = C([0,r];\mathbf{R})$ (as in [11]), i.e.,

$$(\alpha,\beta)_k = \alpha(0)\beta(0) - \int_{-r}^{0}\int_{0}^{\theta} \alpha(\xi-\theta)d\eta_k(\theta)\beta(\xi)d\xi\ , \tag{$1.8_k$}$$

we use the adjoint theory to decompose $C$ by $\Lambda_k := \{\lambda \in \mathbf{C}\ :\ \lambda$ satisfies ($1.3_k$) and $\operatorname{Re}\lambda = 0\}$:

$$C = P_k \oplus Q_k\ , \quad P_k = \operatorname{span}\Phi_k\ , \quad P_k^* = \operatorname{span}\Psi_k\ ,$$
$$(\Psi_k,\Phi_k)_k = I\ , \quad \dim P_k = \dim P_k^* := m_k\ , \dot{\Phi}_k = \Phi_k B_k\ ,$$

where $P_k$ is the generalized eigenspace for ($1.6_k$) associated with $\Lambda_k$ and $B_k$ is an $m_k \times m_k$ constant matrix (throughout this paper, see [8] for standard definitions and results for FDEs).

Similarly to [9], [11], [14], we use the above decompositions to decompose $\mathcal{C}$ by $\Lambda$:

$$\mathcal{C} = P \oplus Q\ , \quad P = \operatorname{Im}\pi\ , \quad \mathcal{Q} = \operatorname{Ker}\pi,$$

where $\dim\, \mathcal{P} = \sum_{k=1}^{N} m_k := M$ and $\pi : \mathcal{C} \longrightarrow \mathcal{P}$ is the projection defined by

$$\pi\phi = \sum_{k=1}^{N} \Phi_k(\Psi_k, \langle\phi(\cdot),\beta_k\rangle)_k\beta_k\ . \tag{1.9}$$

Now, we want to enlarge the phase space $\mathcal{C}$ in such a way that (1.1) can be written as an abstract ODE in a Banach space. To accomplish that, we follow closely the work done for autonomous retarded FDEs in [5].

## 2. Decomposition of the phase space

For phase space, we take the Banach space $BC$ introduced in (H4). In terms of the function $X_0$ above, the elements of $B\mathcal{C}$ have the form $\psi = \phi + X_0\alpha$, with $\phi \in \mathcal{C}, \alpha \in X$, so that $B\mathcal{C} \equiv \mathcal{C} \times X$; its norm is equivalent to the norm $|\phi + X_0\alpha| = |\phi|_{\mathcal{C}} + |\alpha|_X$.

In $B\mathcal{C}$, we consider an extension of the infinitesimal generator of $\{T(t)\}_{t\geq 0}$, still denoted by $A$,

$$A : \mathcal{C}_0^1 \subset B\mathcal{C} \longrightarrow B\mathcal{C}, \quad A\phi = \dot{\phi} + X_0[L(\phi) + d\Delta\phi(0) - \dot{\phi}(0)] \ , \tag{2.1}$$

defined on $\mathcal{C}_0^1 := \{\phi \in \mathcal{C} : \dot{\phi} \in \mathcal{C}, \phi(0) \in \text{dom}\ (\Delta)\}$. On the other hand, $(\cdot,\cdot)_k$ can be continuously defined by the same expression ($1.8_k$) on $C^* \times BC$, where $BC := \{\gamma : [-r,0] \longrightarrow \mathbf{R} \mid \gamma \text{ is continuous on } [-r,0) \text{ and } \exists \lim_{\theta\to 0^-}\gamma(\theta) \in \mathbf{R}\}$. Thus, it is easy to see that $\pi$, as defined in (1.9), is extended to a continuous projection (which we still denote by $\pi$), $\pi : B\mathcal{C} \longrightarrow \mathcal{P}$. In particular, for $\alpha \in X$ we have

$$\pi(X_0\alpha) = \sum_{k=1}^{N} \Phi_k \Psi_k(0)\langle\alpha, \beta_k\rangle\beta_k. \tag{2.2}$$

The projection $\pi$ leads to the topological decomposition

$$B\mathcal{C} = P \oplus \text{Ker}\ \ \pi, \tag{2.3}$$

with the property $\mathcal{Q} \underset{\neq}{\subset} \text{Ker}\,\pi$. With the above notation, we have the following lemma:

**Lemma 2.1.** *$\mathcal{P} \subset \{\phi \in \mathcal{C} : \dot{\phi} \in \mathcal{C}, \phi(0) \in \text{dom}(\Delta), \dot{\phi}(0) = d\Delta\phi(0) + L\phi\}$, and $\pi$ commutes with $A$ in $\mathcal{C}_0^1$.*

*Proof.* From [8], it is known that $P_k$ is contained in the domain of the infinitesimal generator of the $C_0$ semigroup associated with ($1.6_k$); so, $\dot{\phi}(0) = \mu_k\phi(0) + L_k(\phi)$, for $\phi \in P_k, k = 1, \ldots, N$. Therefore, for $\phi = \sum_{k=1}^{N} \Phi_k a_k \beta_k \in \mathcal{P}$, we obtain

$$\begin{aligned} d\Delta\phi(0) + L(\phi) &= \sum_{k=1}^{N}\Big(\Phi_k(0)a_k\mu_k\beta_k + L(\Phi_k a_k\beta_k)\Big) \\ &= \sum_{k=1}^{N}\Big(\Phi_k(0)a_k\mu_k + L_k(\Phi_k a_k)\Big)\beta_k = \sum_{k=1}^{N}\dot{\Phi}_k(0)a_k\beta_k = \dot{\phi}(0), \end{aligned}$$

and the first statement holds. From (H2) and (H3), for $k \in \mathbf{N}$ and $\phi \in \mathcal{C}$ we have

$$\langle L(\phi), \beta_k\rangle\beta_k = \langle L\big(\sum_{j\geq 1}\langle\phi(\cdot), \beta_j\rangle\beta_j\big), \beta_k\rangle\beta_k = L(\langle\phi(\cdot), \beta_k\rangle\beta_k)$$

and, then, from ($1.5_k$),

$$\langle L(\phi), \beta_k\rangle = L_k(\langle\phi(\cdot), \beta_k\rangle). \tag{2.4}$$

Let $\phi \in \mathcal{C}_0^1$. Integrating by parts, using [8, Chap. 7], ($1.7_k$), and ($1.8_k$), we have

$$\begin{aligned}
A(\pi\phi) &= \frac{d}{d\theta}(\pi\phi) = \sum_{k=1}^{N} \dot{\Phi}_k(\Psi_k, \langle\phi(\cdot), \beta_k\rangle)_k\beta_k = \sum_{k=1}^{N} \Phi_k(-\dot{\Psi}_k, \langle\phi(\cdot), \beta_k\rangle)_k\beta_k \\
&= \sum_{k=1}^{N} \Phi_k\Big[-\dot{\Psi}_k(0)\langle\phi(0), \beta_k\rangle + \int_{-r}^{0}\int_{0}^{\theta} \dot{\Psi}_k(\xi-\theta)d\eta_k(\theta)\langle\phi(\xi), \beta_k\rangle d\xi\Big]\beta_k \\
&= \sum_{k=1}^{N} \Phi_k\Big[\Psi_k(0)\int_{-r}^{0} d\eta_k(\theta)\langle\phi(\theta), \beta_k\rangle \\
&\qquad - \int_{-r}^{0}\int_{0}^{\theta} \Psi_k(\xi-\theta)d\eta_k(\theta)\langle\dot{\phi}(\xi), \beta_k\rangle d\xi\Big]\beta_k \\
&= \sum_{k=1}^{N} \Phi_k\Big[\Psi_k(0)\Big(\mu_k\langle\phi(0), \beta_k\rangle + L_k(\langle\phi(\cdot), \beta_k\rangle) - \langle\dot{\phi}(0), \beta_k\rangle\Big) \\
&\qquad + (\Psi_k, \langle\dot{\phi}(\cdot), \beta_k\rangle)_k\Big]\beta_k \ ;
\end{aligned}$$

on the other hand,

$$\begin{aligned}
\pi(A\phi) &= \pi\Big(\dot{\phi} + X_0[L(\phi) + d\Delta\phi(0) - \dot{\phi}(0)]\Big) \\
&= \sum_{k=1}^{N} \Phi_k\Big[(\Psi_k, \langle\dot{\phi}(\cdot), \beta_k\rangle)_k + \Psi_k(0)\langle L(\phi) + d\Delta\phi(0) - \dot{\phi}(0), \beta_k\rangle\Big]\beta_k.
\end{aligned}$$

From (2.4), and since $\langle d\Delta\phi(0), \beta_k\rangle = \mu_k\langle\phi(0), \beta_k\rangle$ by (H2), we conclude that $A(\pi\phi) = \pi(A\phi)$. ☐

Decomposition (2.3) and Lemma 2.1 allow us to decompose (1.1) as a system of abstract ODEs on $\mathbf{R}^M \times \operatorname{Ker}\pi$, with linear and nonlinear parts separated and with finite and infinite dimensional variables also separated in the linear term. More precisely, for $u_t = v(t)$, the abstract ODE in $BC$ associated with (1.1) is

$$\frac{d}{dt}v = Av + X_0F(v). \tag{2.5}$$

Decompose $v \in \mathcal{C}_0^1$ according to (2.3) as $v(t) = \sum_{k=1}^N \Phi_k z_k(t)\beta_k + y(t)$, where $z_k(t) = (\Psi_k, \langle v(t)(\cdot), \beta_k\rangle)_k \in \mathbf{R}^{m_k}, 1 \le k \le N, y(t) \in \mathcal{C}_0^1 \cap \operatorname{Ker}\pi = \mathcal{C}_0^1 \cap \mathcal{Q} := \mathcal{Q}^1$. Then, using (2.2) and Lemma 2.1, we see that in $BC \equiv \mathbf{R}^M \times \operatorname{Ker}\ \pi$ equation (2.5) is equivalent to the system

$$\begin{aligned}
\dot{z}_k &= B_k z_k + \Psi_k(0)\langle F(\sum_{p=1}^{N}\Phi_p z_p\beta_p + y), \beta_k\rangle,\ k = 1, \dots, N, \\
\frac{d}{dt}y &= A_1 y + (I-\pi)X_0F(\sum_{p=1}^{N}\Phi_p z_p\beta_p + y),
\end{aligned} \tag{2.6}$$

where $A_1$ is defined by $A_1 : \mathcal{Q}^1 \subset \operatorname{Ker}\ \pi \longrightarrow \operatorname{Ker}\ \pi, A_1\phi = A\phi$ for $\phi \in \mathcal{Q}^1$, and $\operatorname{Ker}\ \pi$ is taken as a Banach space with the norm induced from $BC$. It will turn out that the spectrum of $A_1$ plays a most important role, so the next lemma is meaningful.

**Lemma 2.2.** $\sigma(A_1) = \sigma_P(A_1) = \sigma(A) \setminus \Lambda$.

*Proof.* First we note that $A_1$ is a closed operator on the Banach space Ker $\pi$, since $A$ is closed as an operator on $BC$. Using the same reasoning as in the proofs of [5, Lemmas (5.1) and (5.2)], we can show that: (i) the operator $A$ from dom $(A)$ to $\mathcal{C}$ and its extension on $BC$ defined by (2.1) have the same spectrum, which is only composed of eigenvalues; (ii) $\sigma_P(A_1) = \sigma(A) \setminus \Lambda$; (iii) $\sigma(A_1) \subset \sigma(A)$. It remains to prove that $(A_1 - \lambda I)$ is surjective if $\lambda \in \Lambda$. Using again the same argument as in the proof of [5, Lemma (5.2)], together with [13, Prop. 4.9], for $\lambda \in \Lambda$ we deduce that Im $(A_1 - \lambda I) \supset \mathcal{Q}$.

Now, consider $f \in \operatorname{Ker} \pi$ written as $f = f^{\mathcal{P}} + f^{\mathcal{Q}} + X_0\alpha$, with $f^{\mathcal{Q}} \in \mathcal{Q}, \alpha \in X$ and $f^{\mathcal{P}} = -\sum_{k=1}^N \Phi_k \Psi_k(0)\langle \alpha, \beta_k\rangle\beta_k$. As $f = (I-\pi)f$, $A$ commutes with $\pi$ in $\mathcal{C}_0^1$ and $\mathcal{C}_0^1 \cap \operatorname{Ker}\ \pi = \mathcal{Q}^1$, then $f \in \operatorname{Im}(A_1 - \lambda I)$ iff $f \in \operatorname{Im}\,(A - \lambda I)$.

Let $\lambda \in \Lambda$. To complete the proof, it is sufficient to show that for each $\alpha \in X$ there exists $h \in \mathcal{C}_0^1$ such that $(A - \lambda I)h = f$, where $f = -\sum_{k=1}^N \Phi_k\Psi_k(0)\langle\alpha,\beta_k\rangle\beta_k + X_0\alpha$; i.e., for each $\alpha \in X$ there exists $h \in \mathcal{C}_0^1$ such that

$$\dot{h} - \lambda h = -\sum_{k=1}^N \Phi_k \Psi_k(0)\langle \alpha, \beta_k\rangle \beta_k, \tag{$2.7_a$}$$

$$L(h) + d\Delta h(0) - \dot{h}(0) = \alpha\ . \tag{$2.7_b$}$$

From $(2.7_a)$, we obtain

$$h(\theta) = e^{\lambda\theta}h(0) - e^{\lambda\theta}\sum_{k=1}^N \Big(\int_0^\theta e^{-\lambda t}\Phi_k(t)\Psi_k(0)dt\Big)\langle\alpha,\beta_k\rangle\beta_k.$$

So, the existence of an $h \in \mathcal{C}_0^1$ satisfying $(2.7_{a,b})$ is equivalent to the existence of $h(0) \in \operatorname{dom}(\Delta)$ such that

$$\begin{aligned}\mathbf{\Delta}(\lambda)h(0) &= -\alpha - \sum_{k=1}^N \Big[L_k\Big(e^{\lambda\theta}\int_0^\theta e^{-\lambda t}\Phi_k(t)\Psi_k(0)dt\Big) - \Phi_k(0)\Psi_k(0)\ \Big]\langle\alpha,\beta_k\rangle\beta_k \\ &= -\alpha + \sum_{k=1}^N (e^{-\lambda \cdot}I_k, \Phi_k)_k \Psi_k(0)\langle\alpha,\beta_k\rangle\beta_k\ ,\end{aligned}$$

where $\mathbf{\Delta}(\lambda)$ is as in (1.2), we write $L(\phi) = L(\phi(\theta))$, and $I_k$ is the $m_k \times m_k$ identity matrix.

Let $h_k, \alpha_k \in \mathbf{R}$, be the coefficients of $h(0), \alpha$, respectively, in $\beta_k$: $h(0) = \sum_{k\geq 1} h_k\beta_k$, $\alpha = \sum_{k \geq 1}\alpha_k\beta_k$. Thus, the former equation is equivalent to the sequence of equations

$$h_k\big(\lambda - \mu_k - L_k(e^{\lambda\cdot})\big) = \begin{cases} -\alpha_k, & \text{if } k > N, \\ -\alpha_k + (e^{-\lambda\cdot}I_k, \Phi_k)_k\Psi_k(0)\alpha_k, & \text{if } 1 \leq k \leq N. \end{cases} \tag{$2.8_k$}$$

For $k > N$, $(2.8_k)$ has a solution $h_k \in \mathbf{R}$, since $\lambda \notin \Lambda$ implies $\lambda - \mu_k - L_k(e^{\lambda\cdot}) \neq 0$. For $1 \leq k \leq N$, the existence of a solution $h_k \in \mathbf{R}$ for $(2.8_k)$ is assured by the last part of the proof of [5, Lemma (5.2)].

It remains to verify that $h(0) \in \operatorname{dom}(\Delta)$. For $p > N$, we define $h^p = \sum_{k=1}^p h_k\beta_k$. Then, $h^p \in \operatorname{dom}(\Delta)$, the sequence $(h^p)$ converges in $X$ to

$$-\alpha + \sum_{k=1}^N (e^{-\lambda\cdot}I_k, \Phi_k)_k\Psi_k(0)\alpha_k$$

and

$$d\Delta(h^p) = \sum_{k=1}^{N} h_k \mu_k \beta_k + \sum_{k=N+1}^{p} (\alpha_k + \lambda h_k - L_k(e^{\lambda\cdot})h_k)\beta_k$$
$$= \sum_{k=1}^{N} h_k \mu_k \beta_k + \sum_{k=N+1}^{p} (\alpha_k + \lambda h_k)\beta_k - L\big(e^{\lambda\cdot} \sum_{k=N+1}^{p} h_k \beta_k\big) \ ,$$

which converges in $X$, as $p \to +\infty$. Since $d\Delta$ is a closed operator, we conclude that $h(0) = \sum_{k\geq 1} h_k \beta_k \in \text{dom }(\Delta)$. □

Before we proceed with the normal form procedure, we write (2.6) in a simpler form by considering its first $N$ equations as a unique equation in $\mathbf{R}^M$. For this, define the $M \times M$ constant matrix $B = \text{diag }(B_1, \dots, B_N)$, the $N \times M$ matrix $\Phi = \text{diag }(\Phi_1, \dots, \Phi_N)$ and the $M \times N$ matrix $\Psi = \text{diag }(\Psi_1, \dots, \Psi_N)$. Then, in $B\mathcal{C} \equiv \mathbf{R}^M \times \text{Ker}\, \pi$, (2.6) becomes

$$\dot{z} = Bz + \Psi(0) \begin{pmatrix} \langle F(\sum_{k=1}^N \Phi_k z_k \beta_k + y), \beta_1 \rangle \\ \dots \\ \langle F(\sum_{k=1}^N \Phi_k z_k \beta_k + y), \beta_N \rangle \end{pmatrix},$$
$$\frac{d}{dt} y = A_1 y + (I - \pi) X_0 F\big(\sum_{k=1}^N \Phi_k z_k \beta_k + y\big), \tag{2.9}$$
$$z = (z_1, \dots, z_N) \in \mathbf{R}^M, y \in \mathcal{Q}^1.$$

## 3. Normal forms for the flow on invariant manifolds

We describe the computation of normal forms using formal series, though we are interested in situations where only a few terms of those series are computed. We consider the formal Taylor expansion

$$F(v) = \sum_{k\geq 2} \frac{1}{j!} F_j(v), \qquad v \in \mathcal{C},$$

where $F_j$ is the $j$th Fréchet derivative of $F$. Then, (2.9) (i.e., (1.1) in $B\mathcal{C} \equiv \mathbf{R}^M \times \text{Ker }\ \pi$) is written as

$$\dot{z} = Bz + \sum_{j\geq 2} \frac{1}{j!} f_j^1(z, y),$$
$$\frac{d}{dt} y = A_1 y + \sum_{j\geq 2} \frac{1}{j!} f_j^2(z, y) \ , \tag{3.1}$$

where $z = (z_1, \dots, z_N) \in \mathbf{R}^M, y \in \mathcal{Q}^1$ and $f_j = (f_j^1, f_j^2), j \geq 2$, are defined by

$$f_j^1(z, y) = \Psi(0) \begin{pmatrix} \langle F_j(\sum_{k=1}^N \Phi_k z_k \beta_k + y), \beta_1 \rangle \\ \dots \\ \langle F_j(\sum_{k=1}^N \Phi_k z_k \beta_k + y), \beta_N \rangle \end{pmatrix},$$
$$f_j^2(z, y) = (I - \pi) X_0 F_j\big(\sum_{k=1}^N \Phi_k z_k \beta_k + y\big) \ . \tag{3.2}$$

As for autonomous FDEs [5], [6], normal forms for (2.9) (or (3.1)) are obtained by a recursive procedure, computing at each step the terms of order $j \geq 2$ from the terms of the same order and the terms of lower orders already computed in previous steps, through a transformation of variables of the form

$$(z,y)=(\hat{z},\hat{y})+\frac{1}{j!}(U_j^1(\hat{z}),U_j^2(\hat{z}))\ , \tag{3.3$_j$}$$

where $z,\hat{z}\in \mathbf{R}^M, y,\hat{y}\in \mathcal{Q}^1$ and $U_j^1:\mathbf{R}^M\longrightarrow \mathbf{R}^M, U_j^2:\mathbf{R}^M\longrightarrow \mathcal{Q}^1$ are homogeneous polynomials of degree $j$ in $z$.

We denote by $\overline{f}_j=(\overline{f}_j^1,\overline{f}_j^2)$ the terms of order $j$ in $(z,y)$ obtained after the computation of the normal form up to order $j-1$, i.e., after performing the change of variables (3.3$_\ell$) of orders $\ell\le j-1$. Following [5, Section 4], we conclude that this recursive process transforms Eq. (2.9) into the equation

$$\begin{aligned}\dot{\overline{z}} &=B\overline{z}+\sum_{j\ge 2}\frac{1}{j!}g_j^1(\overline{z},\overline{y}),\\ \frac{d}{dt}\overline{y} &=A_1\overline{y}+\sum_{j\ge 2}\frac{1}{j!}g_j^2(\overline{z},\overline{y})\ ,\end{aligned} \tag{3.4}$$

where $g_j=(g_j^1,g_j^2)$, $j\ge 2$, are the new terms of order $j$, given by

$$\begin{aligned}g_j^1&=\overline{f}_j^1(z,y)-[DU_j^1(z)Bz-BU_j^1(z)],\\ g_j^2&=\overline{f}_j^2(z,y)-[DU_j^2(z)Bz-A_1(U_j^2(z))],\qquad j\ge 2\ .\end{aligned}$$

Let us introduce the following notation: for a normed space $Y$, $V_j^M(Y)$ denotes the space of homogeneous polynomials of degree $j$ in $M$ variables $z=(z_1,\dots,z_M)$ with coefficients in $Y$,

$$V_j^M(Y)=\Big\{\sum_{|q|=j}c_qz^q\ :\ q\in \mathbf{N}_0^M,\ c_q\in Y\Big\}$$

($z^q=z_1^{q_1}\dots z_M^{q_M}$ for $q=(q_1,\dots,q_M)\in \mathbf{N}_0^M$), with the norm $\big|\sum_{|q|=j}c_qz^q\big|=\sum_{|q|=j}|c_q|_Y$. Defining the operators $M_j=(M_j^1,M_j^2)$, $j\ge 2$, by

$$\begin{aligned}&M_j^1:V_j^M(\mathbf{R}^M)\longrightarrow V_j^M(\mathbf{R}^M),\\ &(M_j^1p)(z)=Dp(z)Bz-Bp(z),\\ &M_j^2:V_j^M(\mathcal{Q}^1)\subset V_j^M(\text{Ker }\pi)\longrightarrow V_j^M(\text{Ker }\pi),\\ &(M_j^2h)(z)=D_zh(z)Bz-A_1(h(z)),\end{aligned} \tag{3.5}$$

and putting $U_j=(U_j^1,U_j^2)$, we have

$$g_j=\overline{f}_j-M_jU_j. \tag{3.6}$$

**Theorem 3.1.** *The operators $M_j^2$, $j\ge 2$, are closed, and their spectra are*

$$\sigma(M_j^2)=\sigma_P(M_j^2)=\{(q,\overline{\lambda})-\mu:\ \mu\in\sigma(A)\setminus\Lambda,\ q\in \mathbf{N}_0^M,|q|=j\},$$

*where $\overline{\lambda}=(\lambda_1,\dots,\lambda_M),\lambda_1,\dots,\lambda_M$ are the elements of $\Lambda$, each one of them appearing as many times as its multiplicity as a root of the associated characteristic equation, $(q,\overline{\lambda})=q_1\lambda_1+\dots+q_M\lambda_M,\ |q|=q_1+\dots+q_M$, for $q=(q_1,\dots,q_M)\in \mathbf{N}_0^M$.*

With a few changes, the proof of this theorem is identical to that of [5, Th. (5.4)], and we omit it. However, one must realize that the main tool for this proof is Lemma 2.2.

This theorem allows us to characterize in spectral terms the situation in which the procedure described above leads to normal forms in a locally center manifold.

**Theorem 3.2.** *Suppose that* (H1)-(H4) *hold and* $\Lambda = \{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda = 0\} \neq \emptyset$. *Then, there exists a formal change of variables* $(z, y) = (\overline{z}, \overline{y}) + O(|\overline{z}|^2)$ *such that:*

(i) *(3.1) is transformed into (3.4), where* $g_j^2(\overline{z}, 0) \equiv 0, j \geq 2$*;*

(ii) *a locally center manifold for (1.1) at zero satisfies* $\overline{y} = 0$*, and the flow on it is given by the* $M$*–dimensional ODE*

$$\dot{\overline{z}} = B\overline{z} + \sum_{j\geq 2} \frac{1}{j!} g_j^1(\overline{z}, 0)\ , \tag{3.7}$$

*which is in normal form (in the usual sense of ODEs).*

*Proof.* From Theorem 3.1 we deduce that $0 \notin \sigma(M_j^2)$, $j \geq 2$. It is then possible to choose $U_j^2$ so that $g_j^2(\overline{z}, 0) \equiv 0$, $j \geq 2$, and (i) follows. The existence of a local center manifold for (1.1) is garanteed by (H1)-(H4). This manifold is tangent to $\mathcal{P}$ at zero. Clearly, for (3.4), $\overline{y} = 0$ is an equation for the center manifold. The transformation of variables can be chosen so that (3.7) is in normal form, since the operators $M_j^1$ are precisely those operators appearing in the computation of normal forms for ODEs in $\mathbf{R}^M$ [1], [3], [7]. □

For the sake of simplicity, we have considered $\Lambda = \{\lambda \in \sigma(A) \ : \mathrm{Re}\,\lambda = 0\}$. Now consider the second case referred to in Section 1, that is, let $\Lambda$ be another nonempty finite subset of $\sigma(A)$. Suppose, as before, that the elements of $\Lambda$ are solutions of the first $N$ equations $(1.3_k)$ and that $\mathcal{P}$ is the invariant space of the linearized equation $\frac{d}{dt}u(t) = d\Delta u(t) + L(u_t)$ associated with $\Lambda$. For this case, we also obtain the results of Section 2, as well as the procedure leading to (3.4) and Theorem 3.1. If there exists a locally invariant manifold $\mathcal{M}_{\Lambda,F}$ for (1.1) tangent to $\mathcal{P}$ at zero, for that manifold we achieve a result similiar to the one stated above for the center manifold, if we guarantee that there are no *resonances* in the second equation of (3.1).

**Definition 3.1.** We say that Eq. (3.1) (or, equivalently Eq. (1.1)) satisfies the **nonresonance conditions relative to** $\Lambda$ if

$$(q, \overline{\lambda}) \neq \mu \quad \text{for all } \mu \in \sigma(A) \setminus \Lambda \text{ and } q \in \mathbf{N}_0^M, |q| \geq 2, \tag{3.8}$$

where $\overline{\lambda} = (\lambda_1, \dots, \lambda_M), \lambda_1, \dots, \lambda_M$ are the elements of $\Lambda$, each one of them appearing as many times as its multiplicity as a root of the associated characteristic equation, $(q, \overline{\lambda}) = q_1\lambda_1 + \cdots + q_M\lambda_M$, $|q| = q_1 + \cdots + q_M$, for $q = (q_1, \dots, q_M) \in \mathbf{N}_0^M$.

*Remark* 3.1. Clearly, if the nonresonance conditions in (3.8) hold, Theorem 3.1 assures that $0 \notin \sigma(M_j^2)$, $j \geq 2$. Thus, $U_j^2$ can be chosen in such a way that $g_j^2(\overline{z}, 0) \equiv 0$, $j \geq 2$, and Theorem 3.2 is still valid if we replace the center manifold by $\mathcal{M}_{\Lambda,F}$, provided that it exists. For instance, if $\Lambda = \{\lambda \in \sigma(A) : \mathrm{Re}\ \lambda > 0\} \neq \emptyset$, then Theorem 3.2 is valid with "unstable manifold" instead of "center manifold". (Note that the existence of this manifold under (H1)-(H4) follows from [11].)

Under (H1)-(H4), we now define normal forms.

**Definition 3.2.** Equation (3.4) is called a **normal form for** (3.1) (or (1.1)) **relative to** $\Lambda$ if $g_j^1, g_j^2$ are defined by (3.6), $U_j^2 = (M_j^2)^{-1}\overline{f}_j^2$, and (3.7) is in normal form.

*Remark* 3.2. In order to have (3.7) in normal form, note that

$$U_j^1(z) = (M_j^1)^{-1} P_j \overline{f}_j^1(z, 0)$$

for $j \geq 2$, where $P_j$ is the projection of $V_j^M(\mathbf{R}^M)$ into Im $M_j^1$ and $(M_j^1)^{-1}$ is a right inverse of $M_j^1$, corresponding to a choice of complementary spaces to Ker $M_j^1$ and Im $M_j^1$ in $V_j^M(\mathbf{R}^M)$ ([5]).

*Remark* 3.3. Consider equations with parameters of type

$$\frac{d}{dt}u(t) = d\Delta u(t) + L(\alpha)(u_t) + F(u_t, \alpha), \tag{3.9}$$

where $\alpha \in \mathbf{R}^p, L : \mathbf{R}^p \longrightarrow \mathcal{L}(C; X), F : \mathcal{C} \times \mathbf{R}^p \longrightarrow X$ are $C^k$ functions, $k \geq 2$, with $F(0, \alpha) = 0, D_1 F(0, \alpha) = 0$ for all $\alpha \in \mathbf{R}^p$. For these equations, the normal form procedure is reduced to that described above by introducing the parameter $\alpha$ as a variable with $\dot{\alpha}(t) = 0$, as done in [6] for FDEs. Given a nonempty finite subset $\Lambda$ of eigenvalues of the infinitesimal generator $A$ associated with the linear PFDE $\frac{d}{dt}u(t) = d\Delta u(t) + L(0)(u_t)$, as in the situation of Remark 3.1, we conclude that the nonresonance conditions relative to $\Lambda$ are now

$$(q, \overline{\lambda}) \neq \mu \quad \text{for all } \mu \in \sigma(A) \setminus \Lambda,\ q \in \mathbf{N}_0^M, |q| \geq 0, \tag{3.10}$$

if the following additional condition is satisfied: $0 \in \Lambda$ whenever $0 \in \sigma(A)$ (cf. [6]).

## 4. The associated FDE

In this section, we associate with Eq. (2.9) (or Eq. (1.1)) an FDE with an equilibrium at zero, in such a way that, under appropriate additional hypotheses, the normal form on the center manifold for that FDE can be chosen so that it coincides with the normal form (3.7) on the center manifold of the original PFDE (1.1), at least up to some finite order. Here, we shall state what happens up to third order only, because we shall apply the results to equations for which a Hopf bifurcation occurs — so, the singularity is generically determined to third order.

Let us first introduce some notation. If there is no possible confusion, we shall use the same symbols to denote corresponding functions, both for the associated FDE to be considered and for (1.1). We denote $C_N := C([-r, 0]; \mathbf{R}^N)$ with the sup norm and define the linear operator $R \in \mathcal{L}(C_N; \mathbf{R}^N)$ by

$$R(\phi) = \big(\mu_k \phi_k(0) + L_k(\phi_k)\big)_{k=1}^N$$

for $\phi = (\phi_1, \dots, \phi_N) \in C_N$. Let $A_0$ be the infinitesimal generator of the $C_0$ semigroup on $C_N$ defined by the solutions of the linear FDE $\dot{x}(t) = R(x_t)$. Its spectrum $\sigma(A_0)$ coincides with its point spectrum $\sigma_P(A_0)$, and $\lambda \in \sigma(A_0)$ iff $\lambda$ satisfies the characteristic equation det $[\lambda I - R(e^{\lambda \cdot} I)] = 0$, which is equivalent to $\lambda - \mu_k - L_k(e^{\lambda \cdot}) = 0$, for some $1 \leq k \leq N$ [8]. Then, $\Lambda = \{\lambda \in \sigma(A) : \mathrm{Re}\,\lambda = 0\}$ is also the set of characteristic eigenvalues of $R$ with real part zero. With the notation of Sections 1 and 2, let $C_N \equiv \prod_{k=1}^N C([-r, 0]; \mathbf{R})$ be decomposed by $\Lambda$:

$$C_N = P \oplus Q \equiv \prod_{k=1}^N (P_k \oplus Q_k),$$

with $P = \prod_{k=1}^N P_k, Q = \prod_{k=1}^N Q_k$. As in [5], [6], we enlarge the space $C_N$, considering the space $BC_N$ of the functions from $[-r, 0]$ to $\mathbf{R}^N$ bounded and continuous on $[-r,0)$ with possibly a jump discontinuity at 0. We decompose $BC_N$ by $\Lambda$ as $BC_N = P \oplus \text{Ker } \pi_0$, where $\pi_0$ is now the projection

$$\pi_0 : BC_N \longrightarrow P, \quad \pi_0(\phi + X_0\alpha) = \Phi[(\Psi,\phi) + \Psi(0)\alpha] \text{ for } \phi \in C_N, \alpha \in \mathbf{R}^N,$$

where $X_0$ is the matrix valued function given by $X_0(\theta) = 0, -r \leq \theta < 0, X_0(0) = I$, and $(\cdot,\cdot)$ is the adjoint bilinear form on $C_N^* \times C_N$ associated with $R$, $(\psi,\phi) = \big((\psi_k,\phi_k)_k\big)_{k=1}^N$, for $\psi = (\psi_k)_{k=1}^N \in C_N^*, \phi = (\phi_k)_{k=1}^N \in C_N$.

**Definition 4.1.** The FDE in $C_N$:

$$\dot{x}(t) = R(x_t) + \Big(\langle F(\sum_{k=1}^N x_{t,k}\beta_k), \beta_p\rangle\Big)_{p=1}^N, \tag{4.1}$$

where $x(t) = \big(x_k(t)\big)_{k=1}^N, x_t = \big(x_{t,k}\big)_{k=1}^N$, is called the **FDE associated with** equation (1.1) **by** $\Lambda$ at zero.

Decomposing $x_t = \Phi z(t) + y_t$, with $z(t) \in \mathbf{R}^M$ and $y_t \in \text{Ker}\,\pi_0 \cap C_N^1 = Q \cap C_N^1 := Q^1$, in $BC_N \equiv \mathbf{R}^M \times \text{Ker } \pi_0$ Eq. (4.1) is equivalent to the system

$$\begin{aligned} \dot{z} &= Bz + \Psi(0)\Big(\langle F\big(\sum_{k=1}^N(\Phi_k z_k + y_k)\beta_k\big), \beta_p\rangle\Big)_{p=1}^N, \\ \frac{d}{dt}y &= A_{0,1}y + (I-\pi_0)X_0\Big(\langle F\big(\sum_{k=1}^N(\Phi_k z_k + y_k)\beta_k\big), \beta_p\rangle\Big)_{p=1}^N, \end{aligned} \tag{4.2}$$

where $A_{0,1} : Q^1 \subset \text{Ker}\,\pi_0 \longrightarrow \text{Ker}\,\pi_0, A_{0,1}\phi = \dot{\phi} + X_0[R(\phi) - \dot{\phi}(0)], \phi \in Q^1$ and $z = (z_1,\ldots,z_N), y = (y_1,\ldots,y_N), z_k \in \mathbf{R}^{m_k}, y_k \in Q_k^1 := Q_k \cap C^1$.

**Theorem 4.1.** *Suppose* (H1)-(H4) *hold and let* $\Lambda = \{\lambda \in \sigma(A) : \text{Re}\,\lambda = 0\} \neq \emptyset$. *Consider* (1.1) *with* $F$ *of class* $C^3$ *given by* $F(v) = \frac{1}{2!}F_2(v) + \frac{1}{3!}F_3(v) + o(|v|^3)$. *With the above notation, assume also*

(H5) $\langle DF_2(u)(\phi\beta_j), \beta_p\rangle = 0$ *for* $1 \leq p \leq N, j > N$ *and all* $u \in \mathcal{P},\ \phi \in C$.

*Then, for a suitable change of variables, the equations on the center manifold for both* (1.1) *and* (4.1) *are the same, up to third order terms.*

The proof of this theorem is based on the possibility of identifying the operators $M_j^2$, $j \geq 2$, defined in (3.5) with the corresponding operators appearing in the computation of normal forms relative to $\Lambda$ for (4.1) [5]. These last operators will be also denoted by $M_j^2$. Throughout this section we always assume (H1)-(H4) and consider $\Lambda = \{\lambda \in \sigma(A) : \text{Re}\,\lambda = 0\} \neq \emptyset$, although we could consider other nonempty finite sets for which conditions (3.8) were satisfied.

To prove Theorem 4.1, we need two preliminary results.

**Definition 4.2.** Let $Y \subset BC_N, Z \subset B\mathcal{C}$. For $F = (F_1,\ldots,F_N) \in V_j^M(Y), f = \sum_{p\geq 1} f_p\beta_p \in V_j^M(Z)$, we say that $F$ coincides with $f$ on the first N coordinates if $F_p = f_p, p = 1,\ldots,N$, and we write $F = f$ $(mod\ \Lambda)$ or $F(z) = f(z)$ $(mod\ \Lambda)$.

**Lemma 4.2.** *For* $j \geq 2$, *let* $f \in V_j^M(\text{Ker}\,\pi), f = \sum_{p\geq 1} f_p\beta_p$ *and suppose that* $h = \sum_{p\geq 1} h_p\beta_p$ *is the solution of equation*

$$M_j^2 h = f \qquad (PFDE)\ . \tag{4.3}$$

*If we define* $F := (f_1, \dots, f_N), H := (h_1, \dots, h_N)$*, then* $F \in V_j^M(\operatorname{Ker}\pi_0)$ *and* $H$ *is the solution of*

$$M_j^2 H = F \qquad (FDE)\ . \tag{4.4}$$

*In other words, if* $f \in V_j^M(\operatorname{Ker}\pi), F = f$ *(mod* $\Lambda$*), then* $F \in V_j^M(\operatorname{Ker}\pi_0)$ *and* $(M_j^2)^{-1}F = (M_j^2)^{-1}f$ *(mod* $\Lambda$*).*

We have written (PFDE), (FDE) in, respectively, (4.3), (4.4) to tell us which operators $M_j^2$ we are considering. We shall do the same below, whenever confusion might occur.

*Proof.* Let $f = \sum_{p\geq 1} f_p\beta_p \in V_j^M(\operatorname{Ker}\pi)$ and $h = \sum_{p\geq 1} h_p\beta_p$ be the unique solution of (4.3) (Theorem 3.1). First, we shall see that $F = (f_1, \dots, f_N) \in V_j^M(\operatorname{Ker}\pi_0)$ and $H = (h_1, \dots, h_N) \in V_j^M(Q^1) = \operatorname{dom}(M_j^2)$ (FDE).

If we write $h_p(z) = \sum_{|q|=j} h_{q,p}z^q$, then we have $h(z) = \sum_{|q|=j} h^q z^q$, where $h^q = \sum_{p\geq 1} h_{q,p}\beta_p$. Since $h \in V_j^M(\mathcal{Q}_1) = \operatorname{dom}(M_j^2)$ (PFDE), we have $h^q \in \mathcal{Q}^1 = \mathcal{C}_0^1 \cap \operatorname{Ker}\pi$. In particular, $h_{q,p}$ are $C^1$ functions and, for $q \in \mathbf{N}_0^M, |q| = j$,

$$\begin{aligned} 0 = \pi(h^q) &= \sum_{p\geq 1} \pi(h_{q,p}\beta_p) = \sum_{p\geq 1} \Big( \sum_{k=1}^N \Phi_k \big(\Psi_k, \langle h_{q,p}(\cdot)\beta_p, \beta_k\rangle\big)_k \beta_k \Big) \\ &= \sum_{k=1}^N \Phi_k\big(\Psi_k, h_{q,k}\big)_k \beta_k, \end{aligned}$$

and then

$$\Phi_k\big(\Psi_k, h_{q,k}\big)_k = 0\ ,\ k = 1, \dots, N,\ |q| = j. \tag{4.5}$$

On the other hand, the above definition of $\pi_0$ yields

$$\pi_0(\phi + X_0\alpha) = \big(\Phi_k(\Psi_k, \phi_k)_k + \Psi_k(0)\alpha_k\big)_{k=1}^N,$$

for $\phi = (\phi_k)_{k=1}^N \in C_N, \alpha = (\alpha_k)_{k=1}^N \in \mathbf{R}^N$. Since $H(z) = \sum_{|q|=j} (h_{q,k})_{k=1}^N z^q$, from (4.5) we obtain $\pi_0\big((h_{q,k})_{k=1}^N\big) = 0$, so $H \in V_j^M(Q^1)$. Similarly, we prove that $F \in V_j^M(\operatorname{Ker}\pi_0)$.

It remains to prove that $H$ is a solution of (4.4). Recall ([5]) that

$$\begin{aligned} (M_j^2 H)(z) &= D_z H(z)Bz - A_{0,1}(H(z)) \\ &= D_z H(z)Bz - \dot{H}(z) + X_0\big[\dot{H}(z)(0) - R(H(z))\big], \end{aligned}$$

where $A_{0,1}$ is as in (4.2) and $\dot{H}(z)$ denotes the derivative of $H(z)(\theta)$ relative to $\theta$. But (4.3) is equivalent to $D_z h(z)Bz - A_1(h(z)) = f(z)$, or, more explicitly, to

$$D_z h_p(z)Bz - \dot{h}_p(z) + X_0\big[\dot{h}_p(z)(0) - \mu_p h_p(z)(0) - L_p(h_p(z))\big] = f_p,\ \forall p \in \mathbf{N}\ , \tag{4.6}$$

where $\dot{h}_p(z)(\theta) = \frac{d}{d\theta} h_p(z)(\theta)$. Therefore, the first $N$ equations of (4.6) are equivalent to (4.4), and the lemma is proved. ☐

Let $F \in C^k, k \geq 2$. We have defined the terms of order $j$ in $(z,y)$, $f_j = (f_j^1, f_j^2), 2 \leq j \leq k$, by (3.2), writing (2.9) as

$$\begin{aligned} \dot{z} &= Bz + \sum_{j=2}^{k} \frac{1}{j!} f_j^1(z,y) + \cdots , \\ \frac{d}{dt}y &= A_1 y + \sum_{j=2}^{k} \frac{1}{j!} f_j^2(z,y) + \cdots , \quad z \in \mathbf{R}^M, y \in \mathcal{Q}^1, \end{aligned} \tag{4.7}$$

where the dots stand for higher order terms. For (4.2) we obtain

$$\begin{aligned} \dot{z} &= Bz + \sum_{j=2}^{k} \frac{1}{j!} f_{0,j}^1(z,y) + \cdots , \\ \frac{d}{dt}y &= A_{0,1} y + \sum_{j=2}^{k} \frac{1}{j!} f_{0,j}^2(z,y) + \cdots , \qquad z \in \mathbf{R}^M, y \in Q^1 , \end{aligned} \tag{4.8}$$

where now

$$\begin{aligned} f_{0,j}^1(z,y) &= \Psi(0)\Big(\langle F_j\big(\sum_{k=1}^{N}(\Phi_k z_k + y_k)\beta_k\big), \beta_p\rangle\Big)_{p=1}^{N}, \\ f_{0,j}^2(z,y) &= (I-\pi)X_0\Big(\langle F_j\big(\sum_{k=1}^{N}(\Phi_k z_k + y_k)\beta_k\big), \beta_p\rangle\Big)_{p=1}^{N}, \end{aligned} \tag{4.9}$$

for $z = (z_1, \dots , z_N), y = (y_1, \dots , y_N), z_k \in \mathbf{R}^{m_k}, y_k \in Q_k^1, k = 1, \dots , N$.

As in Section 3, for (4.2) we define $\overline{f}_{0,j} = (\overline{f}_{0,j}^1, \overline{f}_{0,j}^2), U_{0,j} = (U_{0,j}^1, U_{0,j}^2), g_{0,j} = (g_{0,j}^1, g_{0,j}^2)$ with obvious meanings.

*Remark* 4.1. From (3.2) and (4.9), it is evident that $f_{0,j}^1(z,0) = f_j^1(z,0)$, $j \geq 2$. Since $M_j^1$ defined by (3.5) and the corresponding operators $M_j^1$ appearing in the computation of normal forms for autonomous FDEs are precisely the same ([5]), in particular, for $j = 2$, we deduce that $U_{0,2}^1 = U_2^1$ and $g_{0,2}^1(z,0) = g_2^1(z,0)$ for an adequate choice of variables $(3.3_2)$.

For the coordinates in the infinite dimensional space, we now have the following lemma:

**Lemma 4.3.** *Let $F \in C^2$. Then, $f_{0,2}^2(z,0) = f_2^2(z,0)$ $(mod\ \Lambda), U_{0,2}^2(z) = U_2^2(z)$ $(mod\ \Lambda)$. Futhermore, for $F(v) = \frac{1}{2}F_2(v) + O(|v|^2)$, if* (H5) *is fulfilled, then*

$$D_y f_{0,2}^1(z,y)|_{y=0} U_{0,2}^2(z) = D_y f_2^1(z,y)|_{y=0} U_2^2(z) \ . \tag{4.10}$$

*Proof.* We have

$$\begin{aligned} f_{0,2}^2(z,0) &= (I-\pi_0)X_0\Big(\langle F_2\big(\sum_{k=1}^{N}\Phi_k z_k \beta_k\big), \beta_p\rangle\Big)_{p=1}^{N} \\ &= -\Big(\Phi_p \Psi_p(0)\langle F_2\big(\sum_{k=1}^{N}\Phi_k z_k \beta_k\big), \beta_p\rangle\Big)_{p=1}^{N} + X_0\Big(\langle F_2\big(\sum_{k=1}^{N}\Phi_k z_k \beta_k\big), \beta_p\rangle\Big)_{p=1}^{N}, \end{aligned}$$

and, from (3.2) and (2.2),

$$f_2^2(z,0) = (I-\pi)X_0 F_2\big(\sum_{k=1}^N \Phi_k z_k \beta_k\big)$$

$$= -\sum_{p=1}^N \Phi_p \Psi_p(0)\langle F_2\big(\sum_{k=1}^N \Phi_k z_k\beta_k\big),\beta_p\rangle\beta_p + X_0\sum_{p\geq 1}\langle F_2\big(\sum_{k=1}^N \Phi_k z_k\beta_k\big),\beta_p\rangle\beta_p \ .$$

Then $f_{0,2}^2(z,0) = f_2^2(z,0)$ $(mod\ \Lambda)$. Therefore, since $M_2^2 U_{0,2}^2(z) = f_{0,2}^2(z,0)$ (FDE) and $M_2^2 U_2^2(z) = f_2^2(z,0)$ (PFDE), we conclude from Lemma 4.2 that $U_{0,2}^2(z) = U_2^2(z)$ $(mod\ \Lambda)$.

Suppose now that (H5) holds. Let $h = \sum_{j\geq 1} h_j\beta_j \in \mathcal{Q}^1$ and define $H = (h_k)_{k=1}^N \in Q^1$ (proof of Lemma 4.2). From (4.9) and for the FDE, we have

$$D_y f_{0,2}^1(z,y)|_{y=0}(H) = \Psi(0)\Big(\langle DF_2\big(\sum_{k=1}^N \Phi_k z_k\beta_k\big)\big(\sum_{k=1}^N h_k\beta_k\big),\beta_p\rangle\Big)_{p=1}^N.$$

For the PFDE, and from (3.2) and (H5), we obtain

$$\begin{aligned} D_y f_2^1(z,y)|_{y=0}(h) &= \Psi(0)\Big(\langle DF_2\big(\sum_{k=1}^N \Phi_k z_k\beta_k\big)(h),\beta_p\rangle\Big)_{p=1}^N \\ &= \Psi(0)\Big(\langle DF_2\big(\sum_{k=1}^N \Phi_k z_k\beta_k\big)\big(\sum_{k=1}^N h_k\beta_k\big),\beta_p\rangle\Big)_{p=1}^N \\ &\quad + \Psi(0)\Big(\langle DF_2\big(\sum_{k=1}^N \Phi_k z_k\beta_k\big)\big(\sum_{j>N} h_j\beta_j\big),\beta_p\rangle\Big)_{p=1}^N \\ &= D_y f_{0,2}^1(z,y)|_{y=0}(H)\ . \end{aligned} \tag{4.11}$$

Since $U_{0,2}^2(z) = U_2^2$ $(mod\ \Lambda)$, formula (4.10) is proved. ☐

*Proof of Theorem* 4.1. In [5, Section 6], the following formula is given:

$$\overline{f}_{0,3} = f_{0,3} + \frac{3}{2}[(Df_{0,2})U_{0,2} - (DU_{0,2})g_{0,2}];$$

analogously,

$$\overline{f}_3 = f_3 + \frac{3}{2}[(Df_2)U_2 - (DU_2)g_2].$$

Then, for $y=0$,

$$\overline{f}_{0,3}^1(z,0) = f_{0,3}^1(z,0) + \frac{3}{2}[D_z f_{0,2}^1(z,0)U_{0,2}^1(z) + D_y f_{0,2}^1(z,y)|_{y=0}U_{0,2}^2(z) - DU_{0,2}^1(z)g_{0,2}^1(z,0)],$$

and

$$\overline{f}_3^1(z,0) = f_3^1(z,0) + \frac{3}{2}[D_z f_2^1(z,0)U_2^1(z) + D_y f_2^1(z,y)|_{y=0}U_2^2(z) - DU_2^1(z)g_2^1(z,0)].$$

From Remark 4.1 and Lemma 4.3 we have $\overline{f}_{0,3}^1(z,0) = \overline{f}_3^1(z,0)$. As in Remark 4.1, but now for the case $j=3$ instead of $j=2$, we conclude that $g_{0,3}^1(z,0) = g_3^1(z,0)$ for adequate changes of variables in (4.7) and (4.8), and the theorem is proved. ☐

*Remark* 4.2. As pointed out in Remark 3.1, instead of center manifolds we can consider other invariant manifolds associated with $\Lambda$, provided that they exist and that the nonresonance conditions in (3.8) are satisfied (which is the case for unstable or center–unstable manifolds).

*Remark* 4.3. Certainly, we could find a hypothesis similar to (H5) which would ensure that the equations on the center manifold for (1.1) and (4.1) were the same, up to terms of a certain finite order. However, in the examples presented in the next section, we shall compute only normal forms up to cubic terms.

## 5. Examples

Recall that for PFDEs of type (3.9) the parameter $\alpha$ is introduced as a variable (see Remark 3.3 and [6]) — hence, the role of the variable $z$ is now assumed by $(z, \alpha)$ and $U_j^1, U_j^2$ are functions of $(z, \alpha)$. Note that at the end we can drop the auxiliary equation $\dot{\alpha}(t) = 0$ added to deal with the parameter.

As an illustration, we shall present here two scalar PFDEs of type (3.9) with a generic Hopf singularity at zero (see, e.g., [1], [3], [4], [7], [12] for general work on Hopf bifurcation for ODEs and FDEs). For the first PFDE, (H5) holds and the results of Section 4 are applicable; however, this does not happen for the second PFDE, and then the associated FDE does not provide complete information. In both cases, we shall use complex coordinates, i.e., we shall consider $C = C([-r, 0]; \mathbf{C})$, since complex variables allow us to diagonalize the matrix $B$ in (2.9), which implies that the operators $M_j^1$, $j \geq 2$, are also diagonal, simplifying the computation of the nonlinear terms of the normal form.

**Example 5.1.** Consider the Hutchinson equation with diffusion (see, for instance, [2], [9], [10], [11], [14]):

$$\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} - au(t-1,x)[1+u(t,x)], \quad t > 0, x \in (0,\pi),$$
$$\frac{\partial u(t,x)}{\partial x} = 0\ ,\ x = 0, \pi,$$

where $d > 0, a > 0$. This equation can be written in abstract form in $\mathcal{C} = C([-1,0]; X)$ as

$$\frac{d}{dt}u(t) = d\Delta u(t) + L(a)(u_t) + f(u_t, a), \tag{5.1}$$

where $X = \{v \in W^{2,2}(0,\pi) \ :\ \frac{dv}{dx} = 0 \text{ at } x = 0, \pi\}$, $L(a)(v) = -av(-1), f(v,a) = -av(0)v(-1)$. However, in the following, we shall begin considering the general case of any $f : \mathcal{C} \times \mathbf{R} \to X$ such that $f \in C^3, f(0,a) = 0, D_1 f(0,a) = 0$, for $a > 0$.

The functions

$$\beta_k(x) = \frac{\cos(kx)}{\|\cos(kx)\|_{2,2}}$$

are normalized eigenfunctions of $d\Delta = d(\partial^2/\partial x)$ on $X$, with corresponding eigenvalues $\mu_k = -dk^2, k \geq 0$, and (H1)-(H4) hold for $a > 0$ ([9], [11]). By linearizing (5.1) about the equilibrium $u = 0$, for this case the characteristic equations ($1.3_k$) are

$$\lambda + ae^{-\lambda} + dk^2 = 0, \qquad (k = 0, 1, \dots). \tag{$5.2_k$}$$

[14] showed (using a result of [8]) that, for $a < \pi/2$, all roots of all equations $(5.2_k)$ have negative real parts, so the zero solution is stable; when $a = \pi/2$, $(5.2_0)$ has the unique pair $\pm i\pi/2$ of (simple) solutions on the imaginary axis, and all other solutions of $(5.2_k), k \geq 0$, have negative real parts. To study the qualitative behavior near the critical point $a = \pi/2$, let $a = \pi/2 + \alpha$. Again from [14], there is a pair of solutions $\lambda(\alpha), \overline{\lambda(\alpha)}$ of $(5.2_0)$, with $\lambda(0) = i\pi/2$ and Re $\lambda'(0) > 0$ (Hopf condition), and then a Hopf bifurcation occurs at $\alpha = 0$. With the notation of the preceeding sections (except that now $k \in \mathbf{N}_0$ instead of $k \in \mathbf{N}$), let $\Lambda = \{i\pi/2, -i\pi/2\}$. Defining $L := L(\pi/2)$ and $F(v, \alpha) = -\alpha v(-1) + f(v, \pi/2 + \alpha)$, (5.1) is written as

$$\frac{d}{dt}u(t) = d\Delta u(t) + L(u_t) + F(u_t, \alpha). \tag{5.3}$$

Since $L(\psi\beta_0) = -\frac{\pi}{2}\psi(-1)\beta_0$, the operator $L_0 : C([-1, 0]; \mathbf{R}) \to \mathbf{R}$ corresponding to the eigenvalue $\mu_0 = 0$ and defined by $(1.5_0)$ is $L_0(\psi) = -\frac{\pi}{2}\psi(-1)$. Following Section 4, the FDE associated with (5.1) by $\Lambda$ at the equilibrium point $u = 0, \alpha = 0$ is $\dot{x}(t) = L_0(x_t) + \langle F\big(x_t\beta_0, \alpha\big), \beta_0\rangle$, i.e., the FDE in $C([-1, 0]; \mathbf{R})$

$$\dot{x}(t) = -\frac{\pi}{2}x(t-1) + \langle F(x_t\beta_0, \alpha), \beta_0\rangle. \tag{5.4}$$

We obtain $P = P_0$, dim $\mathcal{P}$ = dim $P_0 = 2$, and $P_0$ = span $\Phi$, where in complex coordinates (cf. [6])

$$\begin{gathered}\Phi(\theta) = (\phi_1(\theta), \phi_2(\theta)) = (e^{i\frac{\pi}{2}\theta}, e^{-i\frac{\pi}{2}\theta}), \quad B = \text{diag }(i\pi/2, -i\pi/2), \\ \Psi(0) = \begin{pmatrix}\psi_1(0)\\ \psi_2(0)\end{pmatrix}, \quad \text{with } \psi_1(0) = \overline{\psi_2(0)} = \frac{1 - i\frac{\pi}{2}}{1 + \frac{\pi^2}{4}}\ .\end{gathered} \tag{5.5}$$

In $B\mathcal{C} = \mathcal{P} \oplus \operatorname{Ker}\pi$ decomposed by $\Lambda$, Eq. (5.3) becomes (cf. (2.9))

$$\begin{aligned}&\dot{z} = Bz + \Psi(0)\langle F(\Phi z\beta_0 + y, \alpha), \beta_0\rangle, \\ &\frac{d}{dt}y = A_1 y + (I - \pi)X_0 F(\Phi z\beta_0 + y, \alpha)\ , \quad z \in \mathbf{C}^2, y \in \mathcal{Q}^1;\end{aligned} \tag{5.6}$$

and, in $BC = P \oplus \operatorname{Ker}\pi_0$, (5.4) becomes (cf. (4.2))

$$\begin{aligned}&\dot{z} = Bz + \Psi(0)\langle F\big((\Phi z + y)\beta_0, \alpha\big), \beta_0\rangle, \\ &\frac{d}{dt}y = A_{0,1}y + (I - \pi_0)X_0\langle F\big((\Phi z + y)\beta_0, \alpha\big), \beta_0\rangle\ , \quad z \in \mathbf{C}^2, y \in Q^1.\end{aligned} \tag{5.7}$$

Suppose that (H5) holds: since $\beta_0 = 1/\sqrt{\pi}$, that means

$$\langle D_1F_2(\Phi c, \alpha)(\psi\beta_k), 1\rangle = 0, \text{ for all } k \in \mathbf{N}, c \in \mathbf{C}^2, \psi \in C.$$

On the other hand, for Eq. (5.3) there exists a two–dimensional local center manifold tangent to $\mathcal{P}$ at $u = 0, \alpha = 0$, which is stable [9]. Theorem 4.1 allows us to conclude that the equations on the center manifold for Eq. (5.3) and Eq. (5.4) (or, equivalently, for Eq. (5.6) and Eq. (5.7)) coincide up to third order terms — which are sufficient to determine a generic Hopf bifurcation. For Eq. (5.4), we are able to write that equation without additional calculus, using [6].

For instance, let Eq. (5.1) be the Hutchinson equation with diffusion:

$$\frac{d}{dt}u(t) = d\Delta u(t) - au(t-1)[1 + u(t)]\ , \tag{5.8}$$

$a, d > 0$. Then $F(v, \alpha) = -\alpha v(-1) - (\frac{\pi}{2} + \alpha)v(-1)v(0)$, and

$$F_2(v, \alpha) = -2\alpha v(-1) - \pi v(-1)v(0), \tag{5.9}$$
$$D_1F_2(v, \alpha)(u) = -2\alpha u(-1) - \pi\big(u(-1)v(0) + v(-1)u(0)\big),$$

and clearly (H5) is fulfilled. In this case, the associated FDE (5.4) in $C$ is

$$\dot{x}(t) = -ax(t-1)[1 + \frac{1}{\sqrt{\pi}}x(t)],$$

which is the well–known Wright equation under the change $x \longmapsto \frac{1}{\sqrt{\pi}}x$, so we can apply the results in [6]. Its equation on the center manifold is given in polar coordinates $(\rho, \xi)$ by

$$\dot{\rho} = \mathrm{Re}\ \lambda'(0)\alpha\rho + \frac{1}{\pi}K\rho^3 + O\big(\alpha^2\rho + |(\rho, \alpha)|^4\big), \tag{5.10}$$
$$\dot{\xi} = \ -\frac{\pi}{2} + O\big(|(\rho, \alpha)|\big)\ ,$$

where $\mathrm{Re}\,\lambda'(0)$ and $K$ are as in [6, Ex. (3.24) with $N = 0$]:

$$\mathrm{Re}\,\lambda'(0) = \frac{2\pi}{4 + \pi^2} > 0, \qquad K = \frac{\pi(2 - 3\pi)}{5(4 + \pi^2)} < 0\ . \tag{5.11}$$

Theorem 4.1 tells us that the flow on the center manifold for (5.8) at $u = 0, a = \pi/2$ is also given by (5.10), where (5.11). Therefore, the periodic solutions associated with the generic Hopf bifurcation for (5.8) are stable, because $K < 0$.

**Example 5.2.** Consider the scalar PFDE (see [13])

$$\frac{\partial u(t, x)}{\partial t} = \frac{\partial^2 u(t, x)}{\partial x^2} + u(t, x) - au(t - 1, x)[1 + u(t, x)], \quad t \geq 0,\ x \in (0, \pi), \tag{5.12}$$
$$u(t, 0) = 0, \quad u(t, \pi) = 0, \quad t \geq 0\ ,$$

where $a > 0$. In the abstract space $\mathcal{C} = C([-1, 0]; X)$, with $X = \{v \in L^2(0, \pi) : v(0) = v(\pi) = 0\}$, this equation is

$$\frac{d}{dt}u(t) = \Delta u(t) + u(t) - au(t - 1) + f(u_t, a), \tag{5.13}$$

for $f : \mathcal{C} \times \mathbf{R} \to X, f(v, a) = -av(-1)v(0)$. But, in the following, we shall consider the general case of any $f \in C^3$ such that $f(0, a) = 0, D_1f(0, a) = 0, a > 0$.

In $X$, the sequence of eigenvalues of $\Delta$ is $\{-k^2\}_{k=1}^{\infty}$, with normalized eigenfunctions $\beta_k(x) = \sqrt{2/\pi}\sin(kx)$. The linearized equation about the equilibrium point zero is

$$\frac{d}{dt}u(t) = \Delta u(t) + L(a)(u_t), \quad \text{for } L(a)(v) := v(0) - av(-1),\ v \in \mathcal{C}\ ,$$

with characteristic equations

$$\lambda + ae^{-\lambda} + (k^2 - 1) = 0 \quad (k = 1, 2, \dots). \tag{5.14$_k$}$$

Using [8, Th.A.5, pg. 339], it can be shown that, for $0 < a < \pi/2$, all roots of (5.14$_k$) have negative real parts, $k \geq 1$. At the critical point $a = \pi/2$, [13] showed that Eq. (5.14$_1$) has two simple roots $\pm i\pi/2$ and the remaining roots have negative real parts; futhermore, all solutions of Eq. (5.14$_k$), $k \geq 2$, have negative real parts. Using [8, Lemma 4.1, pg. 254], we conclude that the Hopf condition is satisfied: changing the parameter $a$ by putting $a = \frac{\pi}{2} + \alpha$, Eq. (5.14$_1$) is $\lambda e^{\lambda} = -\frac{\pi}{2} + \alpha$

and has a pair of eigenvalues $\lambda(\alpha), \overline{\lambda(\alpha)}, \lambda(\alpha) = \gamma(\alpha) + i\sigma(\alpha)$, of class $C^1$, with $\lambda(0) = i\pi/2, \gamma'(0) > 0$; hence, a Hopf bifurcation occurs at $u = 0, \alpha = 0$.

Let $\Lambda = \{i\pi/2, -i\pi/2\}, N = 1$, and recall the notation of the previous sections. Hypotheses (H3)-(H4) hold for $L := L(\pi/2)$. Since $L(\psi\beta_1) = [\psi(0) - \frac{\pi}{2}\psi(-1)]\beta_1$ for $\psi \in C = C([-1,0];\mathbf{R})$, the operator $L_1 : C \to \mathbf{R}$ corresponding to the eigenvalue $\mu_1 = -1$ is

$$L_1(\psi) = \psi(0) - \frac{\pi}{2}\psi(-1),$$

and the associated FDE given by Definition 4.1 has linearization at zero $\dot{x}(t) = -\frac{\pi}{2}x(t-1)$, as for Example 5.1. In $B\mathcal{C} = \mathcal{P} \oplus \operatorname{Ker}\pi$ decomposed by $\Lambda$ and using complex coordinates, Eq. (5.13) is written as (cf. (2.9))

$$\begin{aligned} &\dot{z} = Bz + \Psi(0)\langle F(\Phi z\beta_1 + y, \alpha), \beta_1\rangle, \\ &\frac{d}{dt}y = A_1 y + (I - \pi)X_0 F(\Phi z\beta_1 + y, \alpha), \quad z \in \mathbf{C}^2, y \in \mathcal{Q}^1 , \end{aligned} \tag{5.15}$$

with $B, \Phi = \Phi_1, \Psi(0) = \Psi_1(0)$ still given by (5.5), $\beta_1(x) = \sqrt{\frac{2}{\pi}}\sin x$ and $F(v,\alpha) = -\alpha v(-1) + f(v, \alpha + \pi/2)$, for $v \in \mathcal{C}, \alpha \in \mathbf{R}$. For Eq. (5.13), the associated FDE (4.1) is

$$\dot{x}(t) = -\frac{\pi}{2}x(t-1) + \langle F(x_t\beta_1, \alpha), \beta_1\rangle , \quad x_t \in C.$$

The existence of a two–dimensional local center manifold for Eq. (5.13) tangent to $\mathcal{P}$ at $u = 0, a = \pi/2$ follows from [9]. If $F$ satisfies (H5), then the normal forms on the center manifold for the original PFDE and for the FDE above coincide up to cubic terms, and can be calculated without additional computations using [6].

However, consider the particular case of (5.12), i.e., $f(v,a) = -av(-1)v(0)$. Then, Eq. (5.13) becomes

$$\frac{d}{dt}u(t) = \Delta u(t) + u(t) - au(t-1)[1 + u(t)], \tag{5.16}$$

and its associated FDE by $\Lambda$ is

$$\dot{x}(t) = -ax(t-1)[1 + \frac{4}{3}\Big(\frac{2}{\pi}\Big)^{3/2}x(t)]. \tag{5.17}$$

As in Eq. (5.8), we have (5.9). A few calculations give for $z = (z_1, z_2), \psi \in C, k \geq 2$

$$\langle D_1F_2(\Phi z\beta_1, \alpha)(\psi\beta_k), \beta_1\rangle = \pi[i(z_1 - z_2)\psi(0) - (z_1 + z_2)\psi(-1)]c_k , \tag{5.18}$$

with

$$c_k := \langle \beta_1\beta_k, \beta_1\rangle = \begin{cases} 0, & \text{if } k \text{ even,} \\ -\big(\frac{2}{\pi}\big)^{3/2}\frac{4}{k(k^2-4)}, & \text{if } k \text{ odd.} \end{cases} \tag{5.19}$$

Clearly, (H5) fails, and therefore we cannot apply Theorem 4.1. Nevertheless, we still can profit from the relationship between the PFDE (5.16) and its associated FDE by $\Lambda$ (5.17).

It is well known that, among the cubic terms, it is sufficient to know the coefficients of $\begin{pmatrix} z_1^2 z_2 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ z_1 z_2^2 \end{pmatrix}$ to study the qualitative behavior of the generic Hopf bifurcation.

**Lemma 5.1.** *Let*

$$\dot{z} = Bz + \frac{1}{2}g_2^1(z,0,\alpha) + \frac{1}{3!}g_3^1(z,0,\alpha) + \cdots, \qquad z \in \mathbf{C}^2, \tag{5.20}$$

$$\dot{z} = Bz + \frac{1}{2}g_{0,2}^1(z,0,\alpha) + \frac{1}{3!}g_{0,3}^1(z,0,\alpha) + \cdots, \qquad z \in \mathbf{C}^2, \tag{5.21}$$

*where the dots stand for higher order terms, be normal forms in complex coordinates on the center manifold at zero for (5.16), and (5.17), respectively. Then,*

$$g_3^1(z,0,\alpha) = g_{0,3}^1(z,0,0) + \begin{pmatrix} cz_1^2z_2 \\ \overline{c}z_1z_2^2 \end{pmatrix} + O(|z|\alpha^2),$$

*where*

$$c = \frac{3}{2}\pi^2\psi_1(0)\sum_{k>1}\frac{c_k^2(1+i)}{k^2-1-\frac{\pi}{2}+i\pi},$$

*the bar denotes complex conjugation, and the $c_k$ are given by (5.19).*

*Proof.* Using (4.11) and the proof of Theorem 4.1, for $y=0$ we deduce that

$$\overline{f}_3^1(z,0,\alpha) = \overline{f}_{0,3}^1(z,0,\alpha) + \frac{3}{2}\Psi(0)\langle D_1F_2(\Phi z\beta_1,\alpha)(\sum_{k>1}h_k(z,\alpha)\beta_k),\beta_1\rangle, \tag{5.22}$$

where $h(z,\alpha) := U_2^2(z,\alpha) = \sum_{k\geq 1}h_k(z,\alpha)\beta_k$, i.e., $h(z,\alpha)$ is the unique solution of $(M_2^2h)(z,\alpha) = f_2^2(z,0,\alpha)$. For $h_k(z) := h_k(z,0)$, (5.18) yields

(5.23)
$$\begin{aligned}
&\langle D_1F_2(\Phi z\beta_1,0)(\sum_{k>1}h_k(z)\beta_k),\beta_1\rangle \\
&\quad = \pi\big[i(z_1-z_2)\sum_{k>1}c_kh_k(z)(0) - (z_1+z_2)\sum_{k>1}c_kh_k(z)(-1)\big] \\
&\quad = \pi\big[z_1\sum_{k>1}c_k\big(ih_k(z)(0)-h_k(z)(-1)\big) - z_2\sum_{k>1}c_k\big(ih_k(z)(0)+h_k(z)(-1)\big)\big].
\end{aligned}$$

Now we need to compute $h_k(z)$, by solving the equation $(M_2^2h)(z,0) = f_2^2(z,0,0)$. From (3.2) and (2.2), we have

$$f_2^2(z,0,0) = -\Phi\Psi(0)\langle F_2(\Phi z\beta_1,0),\beta_1\rangle\beta_1 + X_0F_2(\Phi z\beta_1,0),$$

and then $\langle f_2^2(z,0,0),\beta_k\rangle = -\pi X_0\langle\Phi(0)z\Phi(-1)z\beta_1^2,\beta_k\rangle = X_0 i\pi(z_1^2-z_2^2)c_k$ for $k>1$. On the other hand, the definition of $M_2^2$ in (3.5) leads to

$$\begin{aligned}
&D_zh_k(z)Bz - \dot{h}_k(z) = 0, \\
&\dot{h}_k(z)(0) + (k^2-1)h_k(z)(0) + \frac{\pi}{2}h_k(z)(-1) = i\pi(z_1^2-z_2^2)c_k,
\end{aligned} \tag{5.24$_k$}$$

where $k>1$ and $\dot{h}_k(z)(0) = \frac{d}{d\theta}h_k(z)(\theta)|_{\theta=0}$. For each $k>1$, it is easy to solve (5.24$_k$) by setting $h_k(z)(\theta) = \sum_{|q|=2}h_{q,k}(\theta)z^q$. It turns out that $h_{q,k}=0$ for $q\neq(2,0),(0,2)$, $h_{(0,2),k} = \overline{h_{(0,2),k}}$, and, finally,

$$h_k(z)(\theta) = c_k i\pi\Big(\frac{z_1^2}{k^2-1-\frac{\pi}{2}+i\pi}e^{i\pi\theta} - \frac{z_2^2}{k^2-1-\frac{\pi}{2}-i\pi}e^{-i\pi\theta}\Big).$$

Using (5.22) and (5.23), we obtain

$$\overline{f}^1_3(z,0,0)=\overline{f}^1_{0,3}(z,0,0)+\frac{3\pi^2}{2}\Psi(0)\sum_{k>1}c_k^2\Big(\frac{(-1+i)z_1^3+(1+i)z_1^2z_2}{k^2-1-\frac{\pi}{2}+i\pi}+\frac{(1-i)z_1z_2^2-(1+i)z_2^3}{k^2-1-\frac{\pi}{2}-i\pi}\Big).$$

Thus, we write $g^1_3(z,0,0)$ in the form

$$g^1_3(z,0,0)=g^1_{0,3}(z,0,0)+\begin{pmatrix}cz_1^2z_2\\ \overline{c}z_1z_2^2\end{pmatrix},$$

with

$$c=\frac{3}{2}\pi^2\psi_1(0)\sum_{k>1}\frac{c_k^2(1+i)}{k^2-1-\frac{\pi}{2}+i\pi}.$$

Since from [6, Section 3] we know that $g^1_3(z,0,\alpha)\in \mathrm{Ker}\,(M^1_3)$ and $g^1_3(z,0,\alpha)=g^1_3(z,0,0)+O(|z|\alpha^2)$, the lemma is proved. □

Recall that (5.17) is the Wright equation after the change of variables $x\longmapsto c_1x$, with $c_1$ given by (5.19). Hence, we apply to it the results in [6], as in the former example. This, and the lemma above, will give us all the information we need, as we shall describe below.

For the Wright equation $\dot{x}(t)=-ax(t-1)[1+x(t)]$, the normal form on the center manifold at $x=0, a=\frac{\pi}{2}$ ([6, Ex. (3.24) with $N=0$]) is written in polar coordinates $(\rho,\xi)$ as

$$\begin{aligned}\dot{\rho}&=\gamma'(0)\alpha\rho+K_1\rho^3+O\big(\alpha^2\rho+|(\rho,\alpha)|^4\big),\\ \dot{\xi}&=-\frac{\pi}{2}+O\big(|(\rho,\alpha)|\big)\ ,\end{aligned}\tag{5.25}$$

where $K_1=K,\gamma'(0)=\mathrm{Re}\,\lambda'(0)$, and $\mathrm{Re}\,\lambda'(0)$ and $K$ are given by (5.11). Then, (5.21) (that is, the normal form on the center manifold at $x=0, a=\frac{\pi}{2}$ for (5.17)) is given in polar coordinates by the same Eq. (5.25) with $K_1=c_1^2K$. Now, consider (5.20), the normal form on the center manifold for (5.16). From Remark 4.1 (for the variables $z,\alpha$ instead of $z$), we have $g^1_{0,2}(z,0,\alpha)=g^1_2(z,0,\alpha)$. Writing it in real coordinates $w$ through the change of variables $z_1=w_1-iw_2, z_2=w_1+iw_2$, and using Lemma 5.1, as well as the notation used there, we obtain

$$\begin{aligned}\dot{w}=B_1w&+\frac{1}{2}S^{-1}g^1_{0,2}(Sw,0,\alpha)+\frac{1}{3!}S^{-1}g^1_{0,3}(Sw,0,\alpha)\\ &+\frac{1}{3!}\rho^2\begin{pmatrix}(\mathrm{Re}\,c)w_1+(\mathrm{Im}\,c)w_2\\ -(\mathrm{Im}\,c)w_1+(\mathrm{Re}\,c)w_2\end{pmatrix}+O(|w|\alpha^2+|(w,\alpha)|^4),\end{aligned}$$

with $B_1=S^{-1}BS=\begin{pmatrix}0&\frac{\pi}{2}\\ -\frac{\pi}{2}&0\end{pmatrix}, S=\begin{pmatrix}1&-i\\ 1&i\end{pmatrix}$ and $\rho^2=w_1^2+w_2^2$. In polar coordinates $(\rho,\xi)$, we have $w_1=\rho\cos\xi, w_2=\rho\sin\xi$, and since Eq. (5.21) became Eq. (5.25) with $K_1=c_1^2K$, we deduce that this normal form is transformed into

$$\begin{aligned}\dot{\rho}&=\gamma'(0)\alpha\rho+K^*\rho^3+O\big(\alpha^2\rho+|(\rho,\alpha)|^4\big),\\ \dot{\xi}&=\ -\frac{\pi}{2}+O\big(|(\rho,\alpha)|\big)\ ,\end{aligned}\tag{5.26}$$

where $K^* = c_1^2 K + \frac{1}{3!}\mathrm{Re}\, c$ and $\gamma'(0) = \mathrm{Re}\,\lambda'(0)$, $K$ are given by (5.11). Now, we must determine the sign of $K^*$. After a few calculations, we obtain

$$\frac{1}{3!}\mathrm{Re}\, c = \frac{1}{4}\pi^2 \sum_{k>1} c_k^2 \mathrm{Re}\left(\frac{(1+i)\psi_1(0)}{k^2 - 1 - \frac{\pi}{2} + i\pi}\right) = \frac{\pi^2}{4+\pi^2}\sum_{k>1} c_k^2 A_k,$$

with

$$A_k = \frac{(1+\pi/2)(k^2-1-\pi/2) + \pi(1-\pi/2)}{(k^2-1-\pi/2)^2 + \pi^2}.$$

Studying the sign of the derivative of the function $\frac{(1+\pi/2)x+\pi(1-\pi/2)}{x^2+\pi^2}$, we deduce that $A_3 = \max_{k\geq 3} A_k$. On the other hand, Parseval's formula yields

$$\sum_{k>1} c_k^2 = \sum_{k>1} \langle \beta_1^2, \beta_k\rangle^2 = \|\beta_1\|^2 - \langle \beta_1^2, \beta_1\rangle^2 = 1 - c_1^2 = 1 - \frac{2^7}{9\pi^3}.$$

Therefore,

$$K^* < \frac{2^7(2-3\pi) + 5\pi A_3(9\pi^3 - 2^7)}{45\pi^2(4+\pi^2)}.$$

It is straightforward to show that

$$A_3 < \frac{2^7(3\pi - 2)}{5\pi(9\pi^3 - 2^7)};$$

then $K^*$ is negative, and the theorem below follows.

**Theorem 5.2.** *For Eq. (5.16) a generic supercritical Hopf bifurcation occurs from $u = 0, a = \pi/2$ on its center manifold, with the associated periodic solutions being stable.*

*Proof.* Since $K^* < 0$ in (5.26), the periodic solutions are stable in the center manifold, thus stable in the whole space, because there is no root of the characteristic equations ($5.14_k$) with positive real part; and $\gamma'(0)K^* < 0$ implies that the bifurcation is supercritical (cf., e.g., [3], [4], [6]). □

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## References

1. V.I. Arnold, *Geometrical Methods in the Theory of Ordinary Differential Equations*, Springer–Verlag, New York, 1983. MR **84d:**58023
2. S. Busenberg and W. Huang, *Stability and Hopf bifurcation for a population delay model with diffusion effects*, J. Differential Equations **124**(1996), 80–107. MR **97b:**35179
3. S.–N. Chow and J.K. Hale, *Methods of Bifurcation Theory*, Springer–Verlag, New York, 1982. MR **84e:**58019
4. S.–N. Chow and J. Mallet–Paret, *Integral averaging and bifurcation*, J. Differential Equations **26** (1977), 112–159. MR **58:**7718
5. T. Faria and L.T. Magalhães, *Normal forms for retarded functional differential equations and applications to Bogdanov–Takens singularity*, J. Differential Equations **122** (1995), 201–224. MR **97a:**34186a
6. T. Faria and L.T. Magalhães, *Normal forms for retarded functional differential equations with parameters and applications to Hopf bifurcation*, J. Differential Equations **122** (1995), 181–200. MR **97a:**34186b

7. J. Guckenheimer and P. Holmes, *Nonlinear Oscillations, Dynamical Systems and Bifurcations of Vector Fields*, Springer – Verlag, New York, 1983. MR **85f:**58002
8. J.K. Hale, *Theory of Functional Differential Equations*, Springer – Verlag, Berlin, 1977. MR **58:**22904
9. X. Lin, J.W.-H. So and J. Wu, *Centre manifolds for partial differential equations with delays*, Proc. Roy. Soc. Edinburgh **122A** (1992), 237–254. MR **93j:**34116
10. M.C. Memory, *Bifurcation and asymptotic behavior of solutions of a delay–differential equation with diffusion*, SIAM J. Math. Analysis **20** (1989), 533–546. MR **90m:**35029
11. M.C. Memory, *Stable and unstable manifolds for partial functional differential equations*, Nonlinear Anal. **16** (1991), 131–142. MR **92a:**35156
12. A. Stech, *Hopf bifurcation calculations for functional differential equations*, J. Math. Anal. Appl. **109** (1985), 472–491. MR **87h:**58165
13. C.C. Travis and G.F. Webb, *Existence and stability for partial functional differential equations*, Trans. Amer. Math. Soc. **200** (1974), 395–418. MR **52:**3690
14. K. Yoshida, *The Hopf bifurcation and its stability for semilinear diffusion equations with time delay arising in ecology*, Hiroshima Math. J. **12** (1982), 321–348. MR **83m:**58066

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